

# Sparsity-based recovery of three-photon quantum states from two-fold correlations

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Recovery of quantum states from measurements is an essential component in quantum information processing. In quantum optical systems, which naturally offer low decoherence and easy manipulation, quantum states are characterized by correlation measurements. When the states comprise more photons so as to encode more qubits, high-order correlation measurements are required. However, high-order correlations are hard to measure in experiments, as the rate of high-order coincidences decreases very fast when increasing the correlation order. This results in a poor signal-to-noise ratio. Likewise, the number of measurements required to characterize a quantum state increases exponentially with the increase in the number of qubits. Here, we use structure, present in most quantum states of interest (for quantum computing, cryptography, boson sampling, etc.), to recover the full quantum state of three photons from two-fold correlations in a single experimental setup. © 2016 Optical Society of America

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## 1. INTRODUCTION

The field of quantum information has been growing fast over the past decade. The concept of a quantum computer, dating back to Feynman's quantum simulator [1], gained much momentum two decades ago, when Shor proposed a quantum algorithm offering exponential speedup of prime factoring [2]. Ever since, qubits and quantum gates, the basic building blocks of a universal quantum computer, have been proposed and realized experimentally in a variety of physical systems ranging from trapped ions [3] and photons [4] to nuclear magnetic resonance [5]. In particular, optical quantum computation, utilizing the concepts of KLM [6] and cluster states [7], has witnessed experimental realizations of larger and more complex systems in terms of the number of photons [8]. Quantum optical systems, which offer long coherence times and easy manipulation of single qubits and photons, allow probing quantum properties of the light itself [9] and of the physical systems in which it is propagating [10]. In this vein, recently, a linear scheme for quantum computing, relying on the bosonic nature of particles, has been proposed [11] and realized experimentally with up to five photons [9,12–16]. Clearly, the ability to efficiently measure superpositions of quantum states consisting of several photons is essential for the characterization of the prepared states; for the demonstration of computational units, for the operation of certain quantum key distribution protocols [17]—and, finally, it is useful in formulating quantum algorithms [18,19].

To reveal the quantum state, quantum state tomography (QST) is performed. In this process, a measurement corresponding to

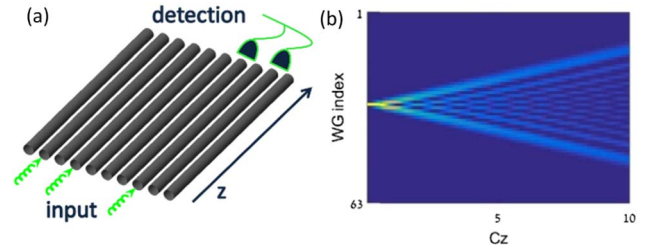
every single element in the density matrix, describing the full quantum state, is repeated many times. However, QST suffers from two main drawbacks: (i) the number of required measurements is very large, and (ii) different physical realizations of measurement systems are necessary. For example, characterizing  $m$  qubits requires  $2^{2m}$  different measurements. In an optical system, this requires rotation of wave plates and polarizers or a complex network of beam splitters and phase shifters. When considering photonic quantum systems, coincidences and correlation functions are being used for characterizing the states. As the number of photons in the system grows, higher-order coincidences and correlation functions are needed [9,12,13,16,20]. However, high-order correlations are hard to measure, as their rate decreases significantly with the order of correlation (see explanation below), which results in a lower signal-to-noise ratio (SNR). Furthermore, as the correlation order increases, more measurements are required.

The issues described above raise a natural question: do low-order correlations hold sufficient information about the high-order correlations to enable the recovery of the complete quantum state? In some special cases—for example, in the context of low-entanglement many-body systems—computing high-order correlations from low-order ones has been recently demonstrated both theoretically [21] and experimentally [22]. Thus, we are motivated to ask, *can a quantum state of  $N$  photons be fully characterized from lower-order correlations?* Furthermore, can this goal be achieved in a single experimental setup, without changing the physical system?

Here, we utilize structure, inherent to many physically interesting quantum states of light, to reduce the complexity in the recovery of a quantum state. We devise a method enabling the recovery of three-photon quantum states (including entangled states) from only two-fold correlation measurements in a single setting. The ability to take two-fold coincidences in a single setup instead of three-fold measurements enables recovery of the quantum states from far fewer measurements and in a considerably higher SNR, due to the fact that two-photon events occur at much greater rates than three-photon ones. For example, in a system with detection quantum efficiency of 10%, for every thousand triplets injected into the system, only one is detected, whereas 27 pairs are recorded within the same acquisition time. This increase in rate improves the SNR of the measurement (in a given integration time), and provides some protection from loss, since not all photons need to be detected. The concept suggested here paves the way to further ideas on structure-based recovery of quantum states, such as recovering a quantum state in an unknown basis in a single setup and recovering the state of several photons without number-resolving detectors.

The concept we devise relies on the fact that, in many cases of interest, the quantum information processed in the system has some characteristic structure (i.e., it is not random). This structure stems from the physical nature of the state and from the fact that, often, we are interested in states that are either pure states or close to pure states (e.g., pure states that have undergone some degradation but are still close to pure states). In this context, having structure in a signal implies that this signal has a sparse representation in some basis; that is, it can be represented in this basis by only a small number of coefficients [23]. Such a signal is then said to be sparse in that basis. For such cases, where the quantum states are close to pure states, we demonstrate the recovery of three-photon states solely from two-fold correlation measurements in a single setup, without the need to change the physical system. We propose a general algorithmic technique to recover the state, provide specific examples with photon-number states and with entangled states, and evaluate the performance of our methodology with respect to sparsity (the number of nonzero eigenvalues of the density matrix) and noise. Finally, we discuss future ideas on how to utilize structure-based concepts even when the sparsity basis is unknown, and related ideas where additional information can be unraveled algorithmically from a partial set of measurements.

We demonstrate the concept of structure-based quantum state recovery on a specific photonic system: an array of  $N_w$  evanescently coupled optical waveguides [Fig. 1(a)]. This system has been used extensively to study fundamental concepts in both the classical and quantum domains (e.g., Bloch oscillations [24,25], Zener tunneling [26], Shockley states [27], bound states in the continuum [28], Anderson localization [29,30], and topological insulators [31], in mean-field limit, and also in the single-photon regime [10,32–34]). In the context presented here, we use this system due to the following reasons: (1) the field forms an inherently discrete set of modes, which is most suitable for quantum information schemes; (2) the waveguide array is lossless and exhibits low decoherence; (3) the system is simple and experimentally realizable; and (4) the waveguides are coupled to each other, which means that measuring light at the output of any waveguide reveals information about other waveguides as well [Fig. 1(b)]. As will be explained below, the spreading of information among the



**Fig. 1.** Physical system and its “quantum impulse response.” (a) Physical system: an array of evanescently coupled optical waveguides. The quantum state is launched at  $z = 0$ . The photons propagate in the array a distance much larger than the coupling length, such that the information is sufficiently spread among the waveguides in the measurement plane. (b) Probability distribution to measure a photon ( $\langle \hat{n}_k(z) \rangle$ ). Here,  $C$  is the coupling constant between adjacent waveguides, when a single photon is injected into the middle waveguide. It serves as the “impulse response” of the system when the number of waveguides approaches infinity.

modes is essential for our approach. A similar technique utilizing measurements at the output facet of such an array of a coupled waveguide to recover the field at the input of the array has been recently demonstrated in the classical realm [35]. It is important to emphasize that, even though we illustrate the idea on this specific system, the concept of structure-based quantum state recovery is completely general and could be implemented in other quantum systems where coupling between the modes exists.

## 2. METHODS

The system is sketched in Fig. 1(a). Photons are injected to the input of the array, are allowed to propagate, and are then measured by detectors at the output facet of the array. The propagation and coupling between the waveguides are modeled by the Hamiltonian [10]

$$H = \beta \sum a_n^\dagger a_n + C \sum (a_{n-1}^\dagger a_n + a_{n+1}^\dagger a_n). \quad (1)$$

Here,  $a_n^\dagger, a_n$  are the creation and annihilation operators in waveguide  $n$ , respectively,  $\beta$  is the propagation constant (identical to all waveguides), and  $C$  is the coupling constant between adjacent waveguides. This Hamiltonian leads to the following Heisenberg equation of motion, which describes the propagation along the  $z$  axis:

$$i\hbar \frac{\partial}{\partial z} a_n^\dagger = -\beta a_n^\dagger - C(a_{n-1}^\dagger + a_{n+1}^\dagger). \quad (2)$$

The simplest case of propagation in the array occurs when the input is strictly into a single waveguide, say, for example, the waveguide at the middle. For this input, the classical solution has a closed form [36], which coincides with the quantum case when a single photon is injected into the middle waveguide [10]. The expectation value of observing that photon after propagating a distance  $z$  (which can be thought of as the “impulse response” of the quantum system for a single photon input) is shown in Fig. 1(b).

Throughout this article, we are interested in recovering an initial quantum state with a fixed number of photons,  $N$ . These states, which are frequently used in most quantum information experiments [8,9], can be generated by parametric downconversion in heralded schemes. For simplicity, consider the case in which the input state consists of three photons. The quantum

state is described by the density matrix of the system. We assume that the basis in which the state is diagonal is known, and this basis will serve as the “sparsity basis.” In general, the sparsity basis can be extracted (learned) from the measurements under certain conditions, or from data with similar features that are often available from other sources [37].

As a first example, consider a state that is diagonal in the Fock basis (another example of an entangled state is given later on). Fock states are often very appealing, both for their nonclassical properties and because common experimental processes [10,13] yield states that are well described by them. The density matrix takes the form

$$\rho = \sum_i p_i |\{n\}^i\rangle \langle \{n\}^i|. \quad (3)$$

Here,  $|\{n\}^i\rangle$  is a Fock state with configuration  $\{n\}^i = \{n_1^i, n_2^i, \dots, n_{N_w}^i\}$ , where the lower index indicates a waveguide number and the upper index  $i$  refers to the  $i$ th configuration (so-called configuration index). This means that for the  $i$ th configuration there are  $n_1^i$  photons in the first waveguide,  $n_2^i$  photons in the second waveguide, and  $n_k^i$  photons in waveguide  $k$ . The coefficient  $p_i$  describes the probability of that configuration to occur. Since the coefficients  $p_i$  are probabilities, they obey  $0 \leq p_i \leq 1$ , in agreement with the general characterization of quantum states. Accordingly, the density operator satisfies  $\rho \geq 0$ . An example of such a state is

$$\rho = p |1_2 1_5 1_7\rangle \langle 1_2 1_5 1_7| + (1-p) |2_3 1_{16}\rangle \langle 2_3 1_{16}|. \quad (4)$$

This density matrix describes the convex sum of two configurations. The first configuration is of probability  $p$ , and it consists of one photon in waveguide 2, one in waveguide 5, and one in waveguide 7, whereas the second configuration is of probability  $1-p$ , and has two photons in waveguide 3, and one photon in waveguide 16. Note that this state consists of three photons.

The problem at hand is to recover the initial state at  $z=0$ , namely the coefficients  $p_i$  [Eq. (3)], from measurements carried out at the output facet after propagating a distance  $z$  in the array, by using photon counting detectors. Generally, characterization of the three-photon state requires three-fold coincidence measurements, which correspond to all possible coefficients of the basis states  $|1_i 1_j 1_k\rangle, |2_i 1_j\rangle, |3_i\rangle$ . These represent the cases where a single photon is launched into each of the waveguides  $i, j, k$ , or where two photons were launched into waveguide  $i$  and one photon into waveguide  $j$ , or where the three photons were all launched into the same waveguide  $i$ , respectively. Such measurements are described by  $\Gamma_{q,r,k}^{(3)} = \text{Tr}(\rho a_q^\dagger a_r^\dagger a_k^\dagger a_k a_r a_q)$ , where  $q, r, k$  are the waveguide indices and the creation and annihilation operators are evaluated at the output facet at distance  $z$ . Instead, in what follows, we will use only two-fold coincidence measurements,  $\Gamma_{q,r}^{(2)} = \text{Tr}(\rho a_q^\dagger a_r^\dagger a_r a_q)$ , which have the advantages described earlier but are missing considerable information. Substituting Eq. (3) into the expression for two-fold coincidences,  $\Gamma_{q,r}^{(2)}$ , we obtain the relation between the probabilities  $p_i$  and the measurements

$$\Gamma_{q,r}^{(2)} = \sum_i p_i |\{n\}^i| a_q^\dagger a_r^\dagger a_r a_q |\{n\}^i|. \quad (5)$$

If we gather the measurements in all the waveguide pairs, casting the problem in a matrix form, then we obtain

$$\mathbf{\Gamma} = M\mathbf{p}. \quad (6)$$

In this formulation,  $\mathbf{\Gamma} \in \mathbb{R}^{N_m}$  holds all the two-fold measurements,  $N_m$  is the number of waveguide pairs,  $\mathbf{p} \in \mathbb{R}^{N_b}$  is the sought coefficient vector (of probabilities),  $N_b$  is the number of basis vectors, and  $M \in \mathbb{R}^{N_m \times N_b}$  is the “sensing matrix” representing the propagation in the array and the relation between the input state and the measurements.

The number of coefficients,  $N_b = \binom{N_w+2}{3}$ , which is the total number of possible configurations, is derived from the number of photons (three, in this case), and the number of waveguides in the system  $N_w$ . Unlike the number of basis vectors, which grows with the number of photons, the number of two-fold coincidences  $N_m = \binom{N_w+1}{2}$  depends only on the number of waveguides. Upon examination of the dimensions of the objects in Eq. (6), we learn that  $N_m < N_b$  always; hence the problem is inherently noninvertible. For example, if we consider three photons in an array of 20 waveguides, then we obtain  $N_m = 210$ , whereas  $N_b = 1540$ . This is a manifestation of using only two-fold coincidences (instead of the three-fold coincidence, which would have made the problem invertible).

To summarize this section, the problem at hand is to find the vector  $\mathbf{p}$  in Eq. (6), which consists of  $N_b$  terms, from the measurement vector  $\mathbf{\Gamma}$ , consisting of  $N_m < N_b$  (real) terms, given the matrix  $M$ . To solve this ill-posed mathematical problem, we need some prior knowledge, which ideally should be rather general. The concept we propose is based on sparsity: the prior knowledge that the initial state has a small number of nonzero elements  $p_i$ , which physically means that the state is close to a pure state. Such states are common in many experimental scenarios. For instance, a pure state subject to local noise results in a low-rank density matrix [38]. Furthermore, whenever a pure state is subject to a low level of “depolarizing noise” (defined as  $\rho \mapsto (1-\lambda)\rho + \frac{\lambda}{N_b}I$  for some  $\lambda \in [0, 1]$ ), it is described by a compressible density matrix. Such a compressible density matrix is not sparse, but it does have one (or several, in the case of almost pure states) significant eigenvalues. Compressible states, albeit not being sparse in the strict sense, do fall under the scope of our method as well. Finally, bipartite states with low enough rank are also appealing theoretically, since they hold a usable entanglement resource, the so-called distillable entanglement [39]. When the basis in which the low-rank state is diagonal is known, the resulting coefficient vector  $\mathbf{p}$  is sparse.

The usage of sparsity has been intensively explored in the field of signal processing, typically under the title of compressed sensing (CS) [23,40]. For classical signals, CS is a field in information science aimed at reducing the number of measurements required for recovering a signal, given that it is sparse in some basis [41]. An essential condition for CS recovery to work well is that each measurement has to carry information; i.e., an impulse input signal should get ‘smeared’ as much as possible in the measurement domain. More recently, CS has been brought into the quantum arena for the purpose of reducing the number of measurements necessary in QST [38] and in quantum process tomography [42], enabling much more efficient tomography. The idea of using sparsity to solve underdetermined inverse problems has opened the door for a wide range of applications in various fields, from sub-Nyquist sampling [43], to subwavelength imaging [44–46],

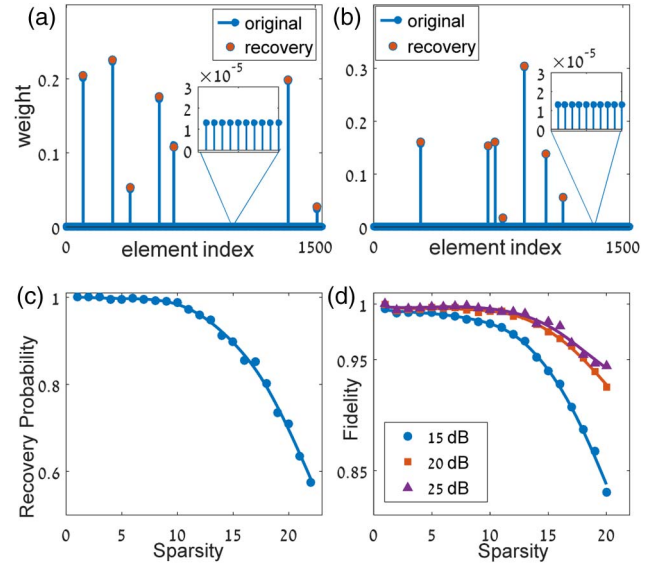
phase retrieval [35,45,47–49], Ankylography [50], holography [51], characterization of incoherent light [52], ghost imaging [53], weak measurements [54,55], measurements of complementary observables [56], and more. To distinguish from these, in this work we use sparsity of the sought quantum state in order to recover a three-photon state from two-fold correlations.

Returning to the problem at hand, we would like to invert Eq. (6): find the vector of probabilities  $\mathbf{p}$  given the measurement vector  $\mathbf{\Gamma}$  (which often also contains noise) and the matrix describing the propagation in the waveguide array  $M$ . In order to overcome the singularity of the problem, we assume that the state is sparse in a known basis, which translates to having a small number of nonzero coefficients  $p_i$ . It is important to stress that we do not need to know their locations or even their number. The only requirement is that there are few in comparison to the total length of the vector. The recovery of the coefficients is performed algorithmically, based on the coupling between the waveguides and propagation in the array. As is known from the field of CS, sparsity-based signal recovery works well if the measurements are carried out in a basis that is least correlated with the basis in which the signal is sparse. It is therefore important to notice that, for sufficiently long propagation (large value of  $Cz$ ), the input signal is smeared by the impulse response of the system [Fig. 1(b)]. This means that performing measurements at the output of a sufficiently long waveguide array facilitates the use of sparsity-based methods.

Our algorithm is based on orthogonal matching pursuit [57], which is commonly used in sparsity-based approaches, with some modifications derived from our constraints. We note that the standard technique of  $l_1$  minimization is not suitable for this problem since we require that every feasible solution has  $l_1$  norm of unity, as  $\mathbf{p}$  is a probability vector. Other common methods that utilize sparsity in various ways are applicable here, such as weighted  $l_1$  (see further detail in Supplement 1).

### 3. RESULTS

Examples of sparsity-based reconstructions in the basis of Fock states are presented in Figs. 2(a) and 2(b). The (simulated) measured data in these examples include SNR of 20 dB Poisson noise in each measurement; that is, the coincidence signal in each measurement has Poissonian statistics with SNR of 20 dB (SNR of 100). In addition, we assume that the original state (the “sought information”) includes 2% depolarizing noise, which simulates many physical cases when the preparation of quantum states is imperfect. Figures 2(a) and 2(b) show the original signal with the bias resulting from the depolarization noise (inset), which makes the signal compressible (see Supplement 1). In the recovery process, we wish to obtain the clean signal (without the bias), which is sparse. In a different scenario, where the characterization of the noise is of interest, the goal may be achieved in an identical setting but using the weighted  $l_1$  minimization procedure. The figures show the original elements of  $\mathbf{p}$  in bars, and the coefficients recovered by our sparsity-based method from two-fold correlations in circles. The number of elements (sparsity) in the original clean signal is seven, as in the recovered one. Thus, our method deals with the compressible signal and recovers a clean (and sparse) one. These examples highlight the fact that our technique enables virtually perfect recovery of three-photon states from two-fold coincidence measurements, in the presence of measurement noise and also even when the original quantum

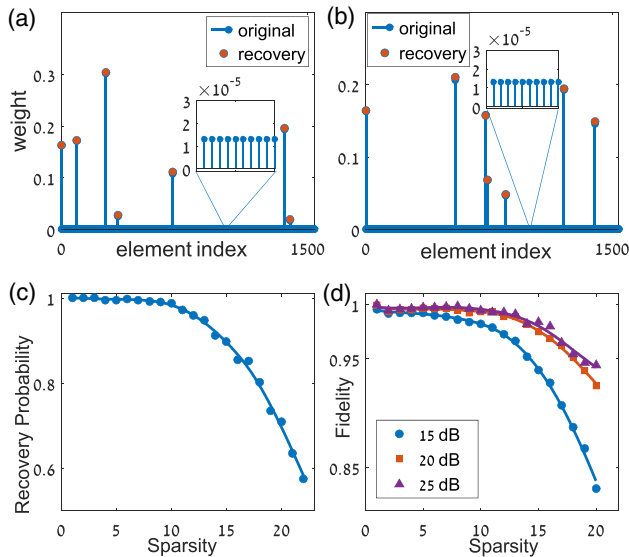


**Fig. 2.** Sparsity-based recovery of Fock states: examples and performance analysis. (a), (b) Examples for the recovery of a state in Fock basis. The original signal has sparsity of 7 (seven nonzero elements in unknown locations and of unknown magnitudes). The state is subject to depolarization noise of 2% (shown in the zoom-in inset). Thus, the state is not sparse, but it is compressible, which allows us to use sparsity in the recovery process. Other types of noise display similar features of compressibility, with similar recovery performance. The “measured data” include the clean measurements at the output facet of the array, with Poisson noise of constant SNR. We wish to recover the true original signal (without the bias noise), which is sparse. As seen in the examples, the recovered signal is indeed practically identical to the true original signal. (c) Recovery probability as a function of the number of nonzero elements in the signal. (d) Dependence of the fidelity of recovery on measurement noise.

state is imperfect. In other words, our sparsity-based method displays robust recovery.

Figure 2(c) shows the recovery probability for different sparsity levels (number of degrees of freedom) in a noiseless scenario. The recovery probability,  $p = \frac{\#(f > 0.95)}{N_r}$ , is defined as the number of recoveries with fidelity higher than 0.95, out of  $N_r = 700$  random realizations of the original quantum state (the signal we wish to recover). The fidelity, defined as  $f = \sum [p_i \hat{p}_i]^{\frac{1}{2}}$ , where  $p_i$  and  $\hat{p}_i$  are the elements of the original and recovered signals, respectively, is evaluated between the recovered state and the original clean state, without the noise. As expected, the recovery probability decreases as the number of nonzero elements in the signal increases; that is, the more sparse the quantum state, the higher the recovery probability. Figure 2(d) shows the performance of our method in terms of fidelity of the two signals in the presence of various noise levels. The same figure also shows the dependence of the fidelity on sparsity. The method works better when the signal is more sparse, but it yields high fidelity recovery (better than 90%) for up to 20 nonzero terms (out of 1540 possible configurations), under 20 dB noise.

We have thus far demonstrated sparsity-based recovery of three-photon states from two-fold coincidence measurements for basis functions that are Fock states. However, the field of quantum information relies heavily on entangled states. It is therefore essential to examine our sparsity-based reconstruction method when the basis includes entangled states. Figure 3 presents exactly that:



**Fig. 3.** Sparsity-based recovery of entangled states: examples and performance analysis. (a), (b) Examples for the recovery of a state in an entangled basis. The original signal has sparsity of 7 (seven nonzero elements in unknown locations and of unknown magnitudes). The state is subject to depolarization noise of 2% (shown in the zoom-in inset). Thus, the state is not sparse, but it is compressible, which allows us to use sparsity in the recovery process. Other types of noise display similar features of compressibility, with similar recovery performance. The “measured data” include the clean measurements at the output facet of the array, with Poisson noise of constant SNR. We wish to recover the true original signal (without the bias noise), which is sparse. As seen in the examples, the recovered signal is indeed practically identical to the true original signal. (c) Recovery probability as a function of the number of nonzero elements in the signal. (d) Dependence of the fidelity of recovery on measurement noise.

Figs. 3(a) and 3(b) show examples where the basis consists of spatially entangled states. As an example, we divide the waveguides between two parties such that “Alice” gets waveguide 7 and the rest of the waveguides belong to “Bob.” The basis now consists of the entangled vectors  $|\psi\rangle = \frac{|2_3 1_7\rangle + |1_3 2_7\rangle}{\sqrt{2}}$ ,  $|\psi^\perp\rangle = \frac{|2_3 1_7\rangle - |1_3 2_7\rangle}{\sqrt{2}}$ , while the rest of the basis terms are Fock states of all the waveguides other than the pair 3,7. A sparse state in this basis is of the form  $\rho = p_1|\psi\rangle\langle\psi| + p_2|\psi^\perp\rangle\langle\psi^\perp| + \sum_{i \neq 1,2} p_i |\{n\}^i\rangle\langle\{n\}^i|$  with a small number of nonzero coefficients  $p_i$ . This basis has 1540 configurations, which is also the number of possible terms in the “sought signal.” In the examples presented in Figs. 3(a) and 3(b), the measurement noise, depolarization noise, and sparsity are the same as in the examples in Fig. 2. The performance of our method, in this basis that includes entangled states, is presented in terms of recovery probability in a noiseless scenario [Fig. 3(c)] and the average fidelity in various noise levels and sparsity values [Fig. 3(d)]. Clearly, our sparsity-based technique, when employed on settings that include entangled states [58], performs as well as it does for the Fock state basis.

It is important to emphasize that using the sparsity-based methodology presented here is conceptual, not specific to a particular algorithm. Our method is based on using very general (generic) prior knowledge—namely, that a state is sparse or compressible—in order to solve a noninvertible problem, which is recovering a three-photon state from two-fold correlations.

Naturally, other algorithms utilizing sparsity could be used to solve the problem, possibly performing even better than ours (see discussion in Supplement 1).

#### 4. CONCLUSION

In conclusion, we showed that prior knowledge in the form of sparsity can be used in order to recover a three-photon state from two-fold correlation measurements. This is achieved by coupling the spatial modes through an array of waveguides, in the spirit of CS, and using a sparsity-based algorithm. The recovery of the quantum state shows high fidelity and excellent robustness to noise (both in the initial state, in the form of depolarization noise, or in the measurement). We also simulated our methodology when the exact number of photons in the state is not known accurately—for example, when the initial state is 90% a three-photon state, and 10% a four- or two-photon state. The performance of our technique remains very high when we use a random coupler (such as the one used in Fig. S2 in Supplement 1), as the initial state is sufficiently sparse (see Supplement 1 for discussion and figure). Finally, the idea of recovering three-photon states from two-fold correlation measurements is readily extendable to recover  $N$ -photon states from  $N - 1$  coincidence measurements, because the mathematics is similar (see Supplement 1 for details). Can this idea be extended to cases in which the measurements are even more incomplete, for example, to recover a four-photon state from two-fold coincidence? We leave that for future work, although our preliminary simulations show that indeed this is possible, with high fidelity and robustness to noise, when the initial states are sufficiently sparse. Moreover, we propose extending the sparsity-based ideas to other, closely related, scenarios. For example, in many experiments, number-resolving photon detectors are needed in order to characterize a state. Such detectors are less available and allow lower detection efficiencies. If we replace the number-resolving detectors with ordinary, simple “bucket” detectors, then the problem becomes noninvertible. Our preliminary results on this problem indicate that sparsity (i.e., having some structure in the sought state) can be used in order to overcome this problem and allow the usage of simple detectors, rendering the usage of number-resolving detectors altogether unnecessary, at least for quantum experiments specifically designed for recovering quantum states.

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See Supplement 1 for supporting content.

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58. Interestingly, some forms of entanglement result in a degeneracy that makes it impossible to distinguish between some of the coefficients. This degeneracy stems from the transfer matrix in some specific bases, and can be anticipated in advance. In such specific cases, the degeneracy can be avoided by using a system that varies in  $z$ . See further details in [Supplement 1](#).